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AVERAGE OF L-FUNCTIONS OF ARTIN-SCHREIER EXTENSIONS

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Abstract. Let $k = \mathbb{F}_q(t)$ be a rational function field over the finite field $\mathbb{F}_q$. In this paper we obtain formulas of average values of L-functions of some family of Artin-Schreier extensions over k.

1. Introduction

The average of a family of L-functions has been studied by many authors. This problem was initiated by Gauss who made two famous conjectures on average values of class numbers of of orders in quadratic fields. These conjectures were proved by Lipschitz in imaginary quadratic fields case and by Siegel [7] in real quadratic fields case. By the Dirichlet's class number formula, these conjectures can be stated as an average of L-functions at $s = 1$ associated to orders in quadratic fields. Takhtadzjan and Vinogradov [8] obtained an average formula for the L-functions of quadratic fields which holds for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 1$. They also gave an average formula for the L-functions of quadratic fields with prime discriminants [9]. Let $k = \mathbb{F}_q(t)$ be a rational function field over the finite field $\mathbb{F}_q$, where q is a power of a prime number p, and $\mathbb{A} = \mathbb{F}_q[t]$ be the polynomial ring. The formulas of average values of the L-functions associated to orders in quadratic extensions of k are obtained by Hoffstein and Rosen [3] when q is odd and by Chen [2] when q is even. Hoffstein and Rosen [3] also gave average formulas for the L-functions associated to maximal orders in quadratic extensions of k. Prime [5] obtained an average of the L-functions associated to maximal orders in ramified imaginary quadratic extensions of k with prime fundamental discriminants. Bae, Jung and Kang [1] obtained averages

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of the L-functions associated to maximal orders of Kummer extensions $K = k(\sqrt[\ell]{P})$ of k, where where ℓ is a prime divisor of $q-1$ and P runs over monic irreducible polynomials in $\mathbb{A}$. Rosen [6] gave averages of the L-functions associated to orders of Kummer extensions of k of degree ℓ. Bae, Jung and Kang [1] obtained averages of the L-functions associated to orders of Artin-Schreier extensions of k. Let $K_u = k(\alpha_u)$ be the Artin-Schreier extension of k generated by a root α_u of $x^p - x = u$, where $u = \frac{B}{A} \in k$ is normalized (see §2.1). Then $G(K_u) = A$ which is a monic polynomial in $\mathbb{A}$ is uniquely determined by the field K_u. In [1], Bae, Jung and Kang gave an average of the L-functions associated to maximal orders of Artin-Schreier extensions K_u of k of degree 2 with monic irreducible $G(K_u)$. In this paper we study the average of the L-functions associated to maximal orders of Artin-Schreier extensions K_u of k of general degree p with monic irreducible $G(K_u)$. In §2, we recall some basic facts on the Artin-Schreier extensions of k and L-functions associated to maximal orders of Artin-Schreier extensions. We also give two key lemmas and their corollaries which play important roles in the computations of average of L-functions. The proofs of these lemmas are given in [1] for $p = 2$. In §3, we give averages of the L-functions associated to maximal orders of real/inert imaginary/ramified imaginary Artin-Schreier extensions K_u of k with monic irreducible $G(K_u)$. In ramified imaginary case, for a given monic irreducible polynomial $P \in \mathbb{A}$, there are infinitely many ramified imaginary Artin-Schreier extensions K_u of k with $G(K_u) = P$, so we also need to fix the degree of numerators of u in the computation of average of L-functions.

2. Preliminaries

2.1. Artin-Schreier extensions

Let $k = \mathbb{F}_q(t)$ and $\mathbb{A} = \mathbb{F}_q[t]$, where q is a power of a prime p. Let $\infty_k = (1/t)$ be the infinite prime of k. We denote by $\mathbb{A}^+$ the set of monic polynomials in $\mathbb{A}$ and by $\mathcal{P}(\mathbb{A})$ the set of monic irreducible polynomials in $\mathbb{A}$. Write $\mathbb{A}_n = \{N \in \mathbb{A} : \deg(N) = n\}$, $\mathbb{A}_n^+ = \mathbb{A}^+ \cap \mathbb{A}_n$ and $\mathcal{P}_n(\mathbb{A}) = \mathcal{P}(\mathbb{A}) \cap \mathbb{A}_n$ $(n \geq 0)$. For any $0 \neq N \in \mathbb{A}$, let $|N| = \#(\mathbb{A}/N\mathbb{A}) = q^{\deg(N)}$, $\Phi(N) = \#(\mathbb{A}/N\mathbb{A})^\times$, where $\#X$ denotes the cardinality of a set X, and $\mathrm{sgn}(N)$ denote the leading coefficient of N. Let $\wp(x) = x^p - x$ be the Artin-Schreier operator. For $u = \frac{B}{A} \in k$ with $A \in \mathbb{A}^+, B \in \mathbb{A}$ and $\gcd(A, B) = 1$, we say that u is *normalized* if it satisfies the following conditions: (i) if $A = \prod_{i=1}^r P_i^{e_i}$, then pe_i for each $1 \leq i \leq r$, (ii) if

$\deg(B) > \deg(A)$, then $p(\deg(B) - \deg(A))$, and (iii) if $\deg(B) = \deg(A)$, then $\mathrm{sgn}(B) \notin \wp(\mathbb{F}_q)$. Let $K_u = k(\alpha_u)$ be the Artin-Schreier extension of k generated by a root α_u of $\wp(\mathbb{F}x) = u$. Let $\mathcal{O}_u$ be the integral closure of $\mathbb{A}$ in K_u. If we write $u = f(T) + \frac{B_1}{A}$ with $f(T) \in \mathbb{A}$ and $\deg(B_1) < \deg(A)$, then one can show that $f(T)$ and A are uniquely determined by the field K_u. Also, if K is an Artin-Schreier extension of k, then there exists such a normalized $u \in k$ such that $K = K_u$. Let $G(K) = A$ be the denominator of u as above. The discriminant d_u of $\mathcal{O}_u$ over $\mathbb{A}$ is $(A \cdot \mathrm{rad}(A))^{p-1}$, where $\mathrm{rad}(A)$ denotes the product of the distinct monic irreducible divisors of A (see [1, Corollary 2.7]). The local discriminant d_{∞_k} at ∞_k is $\infty_k^{(p-1)(\deg(f(T))+1)}$ if $\deg(f(T)) > 0$ and 1 otherwise. The discriminant d_{K_u} of K_u is defined to be $d_u \cdot d_{\infty_k}$. We say that the Artin-Schreier extension K/k is real, inert imaginary or ramified imaginary according as ∞_k splits completely, is inert or ramifies in K. Then, the extension K_u/k is real, inert imaginary or ramified imaginary according as $\deg(B) < \deg(A)$, $\deg(A) = \deg(B)$ or $\deg(A) < \deg(B)$. (See [1, 4] for details.)

2.2. L-functions of Artin-Schreier extensions

Fix an isomorphism $\psi : \mathbb{F}_p \to \mu_p$ sending 1 to a primitive p-th root ζ_p of unity, where μ_p is the group of p-th roots of unity in $\mathbb{C}$. For $u \in k$ and $P \in \mathcal{P}(\mathbb{A})$ which is prime to the denominator of u, define $[u, P) \in \mathbb{F}_p$ by $(P, K_u/k)(\alpha_u) = \alpha_u + [u, P)$, where $(P, K_u/k)$ is the Artin automorphism at P. Extend this to $N \in \mathbb{A}^+$, which is prime to the denominator of u, by multiplicativity. For any $N \in \mathbb{A}^+$, define $\{\frac{u}{N}\}$ to be $\psi([u, N))$ if N is prime to the denominator of u and 0 otherwise. The L-function $L(s, \chi_u^i)$ associated to $\chi_u^i(\cdot) = \{\frac{u}{\cdot}\}^i$ $(0 \le i \le p-1)$ is defined by

$$L(s, \chi_u^i) = \sum_{N \in \mathbb{A}^+} \chi_u^i(N) |N|^{-s}.$$

We can write

$$L(s, \chi_u^i) = \sum_{n=0}^{\infty} \sigma_n^{(i)}(u) q^{-ns} \quad \text{with} \quad \sigma_n^{(i)}(u) = \sum_{N \in \mathbb{A}_n^+} \chi_u^i(N).$$

It is well known that $L(s, \chi_u^i)$ is a polynomial in q^{-s} of degree $\deg(\mathrm{rad}(A)) + \deg(B) - 1$ or $\deg(A) + \deg(\mathrm{rad}(A)) - 1$ according as ∞_k ramifies in K_u or otherwise for $1 \le i \le p-1$.

2.3. Two key lemmas

For $M, N \in \mathbb{A}^+$, two sums $T^{(i)}_{M,N}$ and $\Gamma^{(i)}_{M,N}$ are defined by

$$T^{(i)}_{N,M} = \sum_{\substack{\deg(D)<\deg(M)\\ \gcd(D,M)=1}} \left\{\frac{D/M}{N}\right\}^i, \quad \Gamma^{(i)}_{N,M} = \sum_{\deg(D)<\deg(M)} \left\{\frac{D/M}{N}\right\}^i$$

for $1 \le i \le p-1$. Note that

$$\Gamma^{(i)}_{N,M} = \sum_{\bar{M}\in\mathbb{A}^+, \bar{M}|M} T^{(i)}_{N,\bar{M}}$$

and by Möbius inversion formula,

$$T^{(i)}_{N,M} = \sum_{\bar{M}\in\mathbb{A}^+, \bar{M}|M} \mu(\bar{M})\Gamma^{(i)}_{N,M/\bar{M}}. \tag{2.1}$$

By definition, we have that $T^{(i)}_{N,M} = 0$ if $\gcd(N, M) \ne 1$ and $T^{(i)}_{N,M} = \Phi(M)$ if $\gcd(N, M) = 1$ and N is a p-th power.

Lemma 2.1. *Let $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}^+_n$ with PN. If N is not a p-th power, then $\Gamma^{(i)}_{N,P} = 0$ for $1 \le i \le p-1$.*

Proof. We first consider the case $n \le m$. The set $\{B/P : B \in \mathbb{A}, \deg(B) < m\}$ contains a complete residue system modulo N. So the map $B \mapsto \{\frac{B/P}{N}\}^i$ is a surjective additive character from $\{B \in \mathbb{A} : \deg(B) < m\}$ onto μ_p. Hence $\Gamma^{(i)}_{N,P}$ is a multiple of $1+\zeta_p+\cdots+\zeta_p^{p-1} = 0$, i.e, $\Gamma^{(i)}_{N,P} = 0$. Now assume $n > m$, say $n = m + h$ for some positive integer h. Since $\{B/P : B \in \mathbb{A}, \deg(B) < n\}$ contains a complete residue system modulo N, we have

$$0 = \sum_{\deg(B)<n} \left\{\frac{B/P}{N}\right\}^i = \Gamma^{(i)}_{N,P} + \sum_{l=0}^{h-1}\sum_{B\in\mathbb{A}_{m+l}} \left\{\frac{B/P}{N}\right\}^i$$

as above. For any $B \in \mathbb{A}_{m+l}$, we can write $B = QP + R$ with $Q \in \mathbb{A}_l$ and $\deg(R) < m$. Then,

$$\sum_{B\in\mathbb{A}_{m+l}} \left\{\frac{B/P}{N}\right\}^i = \Big(\sum_{Q\in\mathbb{A}_l} \left\{\frac{Q}{N}\right\}^i\Big)\Big(\sum_{\deg(R)<m} \left\{\frac{R/P}{N}\right\}^i\Big) = \Gamma^{(i)}_{N,P}\sum_{Q\in\mathbb{A}_l} \left\{\frac{Q}{N}\right\}^i.$$

Hence, we get

$$0 = \Gamma^{(i)}_{N,P} + \Gamma^{(i)}_{N,P}\sum_{l=0}^{h-1}\sum_{Q\in\mathbb{A}_l} \left\{\frac{Q}{N}\right\}^i = \Gamma^{(i)}_{N,P}\Big(1 + \sum_{\deg(Q)<h} \left\{\frac{Q}{N}\right\}^i\Big).$$

Assume that $\Gamma^{(i)}_{N,P} \neq 0$. Then $\sum_{\deg(Q)<h}\{\frac{Q}{N}\}^i = -1$. If there exists $Q \in \mathbb{A}$ with $\deg(Q) < h$ such that $\{\frac{Q}{N}\} \neq 1$, then $\sum_{\deg(Q)<h}\{\frac{Q}{N}\}^i = 0$, which is a contradiction. But, if $\{\frac{Q}{N}\} = 1$ for all $Q \in \mathbb{A}$ with $\deg(Q) < h$, then $\sum_{\deg(Q)<h}\left\{\frac{Q}{N}\right\}^i = q^h$, which is also a contradiction. Therefore, we have $\Gamma^{(i)}_{N,P} = 0$. $\square$

Corollary 2.2. *Let $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}^+_n$ with PN. If N is not a p-th power, then $T^{(i)}_{N,P} = -1$ for $1 \le i \le p-1$.*

Proof. By Lemma 2.1 and (2.2), we have $T^{(i)}_{N,P} = -\Gamma^{(i)}_{N,1} = -1$. $\square$

For $M, N \in \mathbb{A}^+$ and positive integer c, two sums $\tilde{T}^{(i)}_{N,M,c}$ and $\tilde{\Gamma}^{(i)}_{N,M,c}$ are defined by

$$\tilde{T}^{(i)}_{N,M,c} = \sum_{\substack{\deg(B)=\deg(M)+c \\ \gcd(B,M)=1}} \left\{\frac{B/M}{N}\right\}^i, \quad \tilde{\Gamma}^{(i)}_{N,M,c} = \sum_{\deg(B)=\deg(M)+c} \left\{\frac{B/M}{N}\right\}^i$$

for $1 \le i \le p-1$. Note that

$$\tilde{\Gamma}^{(i)}_{N,M,c} = \sum_{\bar{M}\in\mathbb{A}^+, \bar{M}|M} \tilde{T}^{(i)}_{N,\bar{M},c}$$

and by Möbius inversion formula,

$$\tilde{T}^{(i)}_{N,M,c} = \sum_{\bar{M}\in\mathbb{A}^+, \bar{M}|M} \mu(\bar{M})\tilde{\Gamma}^{(i)}_{N,M/\bar{M},c}. \tag{2.2}$$

By definition, we have that $\tilde{T}^{(i)}_{N,M,c} = 0$ if $\gcd(N, M) \neq 1$ and $\tilde{T}^{(i)}_{N,M,c} = (q-1)q^h\Phi(M)$ if $\gcd(N, M) = 1$ and N is p-th power.

Lemma 2.3. *Let $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}^+_n$ with PN. If N is not a p-th power, then $\tilde{\Gamma}^{(i)}_{N,P,c} = 0$ for $1 \le i \le p-1$.*

Proof. Since $\Gamma^{(i)}_{N,P} = 0$, there exists $B_0 \in \mathbb{A}$ with $\deg(B_0) < m$ such that $\{\frac{B_0/P}{N}\} \neq 1$, say $\{\frac{B_0/P}{N}\} = \zeta_p^{j_0}$ for some $1 \le j_0 \le p-1$. Let X_a be the set of $B \in \mathbb{A}_{m+c}$ such that $\{\frac{B/P}{N}\}^i = \zeta_p^a$ $(0 \le a \le p-1)$. Let j_a be an integer such that $(ij_0)j_a \equiv a \bmod p$. Then the map $B \mapsto B + j_aB_0$ is a bijection from X_0 onto X_a. Hence, we have

$$\tilde{\Gamma}^{(i)}_{N,P,c} = \sum_{a=0}^{p-1}\sum_{B\in X_a} \left\{\frac{B/P}{N}\right\}^i = |X_0|(1 + \zeta_p + \cdots + \zeta_p^{p-1}) = 0,$$

which completes the proof. □

Corollary 2.4. *Let $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}_n^+$ with PN. If N is not a p-th power, then*

$$\tilde{T}^{(i)}_{N,P,c} = -\sum_{B \in \mathbb{A}_c} \left\{ \tfrac{B}{N} \right\}^i$$

for $1 \leq i \leq p-1$.

Proof. It follows from Lemma 2.3 and (2.2). □

3. Average of L-functions of Artin-Schreier extensions

Let $\zeta_{\mathbb{A}}(s) = \sum_{N \in \mathbb{A}^+} |N|^{-s}$ be the zeta function of $\mathbb{A}$. It is easy to see that $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$. In this section we study the averages of L-functions associated to maximal orders of real/inert imaginary/ramified imaginary Artin-Schreier extensions K_u of k with monic irreducible $G(K_u)$, respectively.

3.1. Real case

For $P \in \mathcal{P}(\mathbb{A})$, let $\mathfrak{F}_P = \{B \in \mathbb{A} : B \neq 0, \deg(B) < \deg(P)\}$ and $\mathcal{F}_P$ be the set of real Artin-Schreier extensions K of k with $G(K) = P$. It is easy to show that for any $B_1, B_2 \in \mathfrak{F}_P$, $K_{B_1/P} = K_{B_2/P}$ if and only if $B_1 = B_2$. Hence, the map $B \mapsto K_{B/P}$ is a bijection from $\mathfrak{F}_P$ onto $\mathcal{F}_P$.

Theorem 3.1. *For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ and $1 \leq i \leq p-1$, we have*

$$\lim_{m \to \infty} \frac{\sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{F}_P} L(s, \chi^i_{B/P})}{(q^m - 1)\#\mathcal{P}_m(\mathbb{A})} = \zeta_{\mathbb{A}}(ps).$$

Proof. Let

$$f_m(s) = \frac{\sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{F}_P} L(s, \chi^i_{B/P})}{(q^m - 1)\#\mathcal{P}_m(\mathbb{A})} - \zeta_{\mathbb{A}}(ps).$$

Since $L(s, \chi^i_{B/P})$ is a polynomial in q^{-s} of degree $2m-1$ for $P \in \mathcal{P}_m(\mathbb{A})$ and $B \in \mathfrak{F}_P$, we have

(3.1)
$$f_m(s) = \frac{\sum_{n=0}^{2m-1} \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{F}_P} \sigma^{(i)}_n(B/P) q^{-ns}}{(q^m - 1)\#\mathcal{P}_m(\mathbb{A})} - \sum_{n=0}^{\infty} q^{(1-ps)n}.$$

Put

$$f_{m,n} = \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{F}_P} \sigma^{(i)}_n(B/P).$$

Then, we have

$$f_{m,n} = \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{N \in \mathbb{A}_n^+} \sum_{B \in \mathfrak{F}_P} \left\{\frac{B/P}{N}\right\}^i = \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{N \in \mathbb{A}_n^+} T_{N,P}^{(i)}.$$

By definition, we have that $T_{N,P}^{(i)} = 0$ if $P|N$ and $T_{N,P}^{(i)} = q^m - 1$ if PN and N is a p-th power. If PN and N is not a p-th power, by Corollary 2.2, we have $T_{N,P}^{(i)} = -1$. For pn, since any $N \in \mathbb{A}_n^+$ will never be a p-th power, we have

$$f_{m,n} = \begin{cases} -q^n \# \mathcal{P}_m(\mathbb{A}) & \text{if } n < m, \\ -(q^n - q^{n-m}) \# \mathcal{P}_m(\mathbb{A}) & \text{if } m \leq n \leq 2m-1. \end{cases} \tag{3.2}$$

For $p|n$, we have

$$f_{m,n} = (q^m - 1) \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N: p\text{-th power}}} 1 - \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N: \text{not } p\text{-th power}}} 1. \tag{3.3}$$

For $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}_n^+$, since $n \leq 2m - 1$, if N is a p-th power, then N is not divisible by P. Hence we have

$$\sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N: p\text{-th power}}} 1 = \sum_{\substack{N \in \mathbb{A}_n^+ \\ N: p\text{-th power}}} 1 = q^{\frac{n}{p}} \tag{3.4}$$

and

$$\sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N: \text{not } p\text{-th power}}} 1 = \begin{cases} q^n - q^{\frac{n}{p}} & \text{if } n < m, \\ q^n - q^{\frac{n}{p}} - q^{n-m} & \text{if } m \leq n \leq 2m-1. \end{cases} \tag{3.5}$$

For $p|n$, by inserting (3.4) and (3.5) into (3.3), we have

$$f_{m,n} = q^{\frac{n}{p}} (q^m - 1) \# \mathcal{P}_m(\mathbb{A}) - \begin{cases} (q^n - q^{\frac{n}{p}}) \# \mathcal{P}_m(\mathbb{A}) & \text{if } n < m, \\ (q^n - q^{\frac{n}{p}} - q^{n-m}) \# \mathcal{P}_m(\mathbb{A}) & \text{if } m \leq n \leq 2m-1. \end{cases} \tag{3.6}$$

By inserting (3.2) and (3.6) into (3.1) and rearranging the terms, we have

$$\begin{aligned} f_m(s) = &- \sum_{n = [\frac{2m-1}{p}]+1}^{\infty} q^{(1-ps)n} \\ &- \frac{1}{q^m - 1} \left(\sum_{n=0}^{2m-1} q^{(1-s)n} - \sum_{n=0}^{[\frac{2m-1}{p}]} q^{(1-ps)n} - q^{-m} \sum_{n=m}^{2m-1} q^{(1-s)n} \right). \end{aligned}$$

For $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{p}$, since $1 - p\sigma < 0$, we have

$$\Bigg|\sum_{n=[\frac{2m-1}{p}]+1}^{\infty} q^{(1-ps)n}\Bigg| \leq \sum_{n=[\frac{2m-1}{p}]+1}^{\infty} q^{(1-p\sigma)n} = \frac{q^{(1-p\sigma)([\frac{2m-1}{p}]+1)}}{1-q^{1-p\sigma}} \to 0$$

and

$$\left|\frac{\sum_{n=0}^{[\frac{2m-1}{p}]} q^{(1-ps)n}}{q^m-1}\right| \leq \frac{\sum_{n=0}^{[\frac{2m-1}{p}]} q^{(1-p\sigma)n}}{q^m-1} < \frac{[\frac{2m-1}{p}]+1}{q^m-1} \to 0$$

as $m \to \infty$. Now, for $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{2}$, we have

$$\left|\frac{\sum_{n=0}^{2m-1} q^{(1-s)n}}{q^m-1}\right| \leq \frac{\sum_{n=0}^{2m-1} q^{(1-\sigma)n}}{q^m-1}.$$

If $\sigma \neq 1$,

$$\frac{\sum_{n=0}^{2m-1} q^{(1-\sigma)n}}{q^m-1} = \frac{q^{-m}-q^{m(1-2\sigma)}}{(1-q^{-m})(1-q^{(1-\sigma)})} \to 0$$

and if $\sigma = 1$,

$$\frac{\sum_{n=0}^{2m-1} q^{(1-\sigma)n}}{q^m-1} = \frac{2m}{q^m-1} \to 0$$

as $m \to \infty$. Finally, for $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{2}$, we have

$$\left|\frac{\sum_{n=m}^{2m-1} q^{(1-s)n}}{q^m(q^m-1)}\right| \leq \frac{\sum_{n=m}^{2m-1} q^{(1-\sigma)n}}{q^m(q^m-1)} \leq \frac{\sum_{n=0}^{2m-1} q^{(1-\sigma)n}}{q^m-1} \to 0$$

as $m \to \infty$. Therefore, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$, we get $f_m(s) \to 0$ as $m \to \infty$, which completes the proof. □

3.2. Inert imaginary case

Let $\{0, \xi_1, \ldots, \xi_{p-1}\}$ be a set of representatives of $\mathbb{F}_q/\wp(\mathbb{F}_q)$. For $P \in \mathcal{P}(\mathbb{A})$, let $\mathfrak{G}_P = \{\xi_a P + B : B \in \mathfrak{F}_P, 1 \leq a \leq p-1\}$ and $\mathcal{G}_P$ be the set of inert imaginary Artin-Schreier extensions K of k with $G(K) = P$. It is easy to show that, for any $B_1, B_2 \in \mathfrak{F}_P$ and $1 \leq a, b \leq p-1$, $K_{(\xi_a P+B_1)/P} = K_{(\xi_b P+B_2)/P}$ if and only if $a = b$ and $B_1 = B_2$. Thus, the map $\xi_a P + B \mapsto K_{(\xi_a P+B)/P}$ is a bijection from $\mathfrak{G}_P$ onto $\mathcal{G}_P$.

THEOREM 3.2. *For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ and $1 \leq i \leq p-1$, we have*

$$\lim_{m\to\infty} \frac{\sum_{B\in\mathfrak{F}_P}\sum_{a=1}^{p-1} L(s, \chi^i_{\xi_a+B/P})}{(p-1)(q^m-1)\#\mathcal{P}_m(\mathbb{A})} \sum_{P\in\mathcal{P}_m(\mathbb{A})} = \zeta_{\mathbb{A}}(ps).$$

Proof. Let

$$g_m(s) = \frac{\sum_{B\in\mathfrak{F}_P}\sum_{a=1}^{p-1} L(s, \chi^i_{\xi_a+B/P})}{(p-1)(q^m-1)\#\mathcal{P}_m(\mathbb{A})} \sum_{P\in\mathcal{P}_m(\mathbb{A})} -\zeta_{\mathbb{A}}(ps).$$

Since $L(s,\chi^i_{\xi_a+B/P})$ is a polynomial in q^{-s} of degree $2m-1$ for any $P\in\mathcal{P}_m(\mathbb{A})$ and $\xi_aP+B\in\mathfrak{G}_P$, we have

$$g_m(s)=\frac{\sum_{n=0}^{2m-1}\sum_{P\in\mathcal{P}_m(\mathbb{A})}\sum_{B\in\mathfrak{F}_P}\sum_{a=1}^{p-1}\sigma_n^{(i)}(\xi_a+B/P)q^{-ns}}{(p-1)(q^m-1)\#\mathcal{P}_m(\mathbb{A})}-\sum_{n=0}^{\infty}q^{(1-ps)n}.$$

Put

$$g_{m,n}=\sum_{P\in\mathcal{P}_m(\mathbb{A})}\sum_{B\in\mathfrak{F}_P}\sum_{a=1}^{p-1}\sigma_n^{(i)}(\xi_a+B/P).$$

Then we have

$$\begin{aligned}g_{m,n}&=\sum_{P\in\mathcal{P}_m(\mathbb{A})}\sum_{B\in\mathfrak{F}_P}\sum_{a=1}^{p-1}\sum_{N\in\mathbb{A}_n^+}\left\{\tfrac{(\xi_aP+B)/P}{N}\right\}^i\\&=\sum_{P\in\mathcal{P}_m(\mathbb{A})}\sum_{N\in\mathbb{A}_n^+}\sum_{a=1}^{p-1}\left\{\tfrac{\xi_a}{N}\right\}^iT_{N,P}^{(i)}.\end{aligned}$$

Since $\wp(\mathbb{F}_q)$ is contained in the kernel of $Tr_{\mathbb{F}_q/\mathbb{F}_p}:\mathbb{F}_q\to\mathbb{F}_p$, we have $\{Tr_{\mathbb{F}_q/\mathbb{F}_p}(\xi_a):1\le a\le p-1\}=\mathbb{F}_p\backslash\{0\}$. Note that $\{\frac{\xi_a}{N}\}=\psi(Tr_{\mathbb{F}_q/\mathbb{F}_p}(\xi_a))^n$ for $N\in\mathbb{A}_n^+$. If pn, then $\sum_{a=1}^{p-1}\{\frac{\xi_a}{N}\}^i=\zeta_p+\cdots+\zeta_p^{p-1}=-1$, so $g_{m,n}=-f_{m,n}$. If $p|n$, then $\{\frac{\xi_a}{N}\}=1$, so $g_{m,n}=(p-1)f_{m,n}$. For the rest of proof, we can now follow a similar procedure in the proof of Theorem 3.1 to show $g_m(s)\to 0$ as $m\to\infty$. □

3.3. Ramified imaginary case

For $P\in\mathcal{P}(\mathbb{A})$ and positive integer c with pc, let $\mathfrak{H}_{P,c}=\{B\in\mathbb{A}: PB,\deg(B)=\deg(P)+c\}$ and $\mathcal{H}_{P,c}$ be the set of ramified Artin-Schreier extensions K of k with $G(K)=P$ and whose discriminant d_K is $P^{2(p-1)}\cdot\infty_k^{(p-1)(c+1)}$. It is easy to show that, for any $B,B'\in\mathfrak{H}_{P,c}$, we have that $K_{B/P}=K_{B'/P}$ if and only if $B'=B+P(D^p-D)$ for some $D\in\mathbb{A}$. We say that $B,B'\in\mathfrak{H}_{P,c}$ are equivalent, denoted by $B\sim B'$, if $B'=B+P(D^p-D)$ for some $D\in\mathbb{A}$. Let $[B]$ be the equivalence class of $B\in\mathfrak{H}_{p,c}$ with respect to $\sim$, and $\tilde{\mathfrak{H}}_{P,c}=\{[B]:B\in\mathfrak{H}_{P,c}\}$. Then, the map $[B]\mapsto K_{B/P}$ is a bijection from $\tilde{\mathfrak{H}}_{P,c}$ onto $\mathcal{H}_{P,c}$. For $B\in\mathfrak{H}_{P,c}$, we have a surjective map

$$\{D\in\mathbb{A}:\deg(D)\le[c/p]\}\to[B],\ D\mapsto B+P(D^p-D). \tag{3.7}$$

For $D,E\in\mathbb{A}$ with $\deg(D),\deg(E)\le[c/p]$, we have that $B+P(D^p-D)=B+P(E^p-E)$ if and only if $D-E\in\mathbb{F}_p$. Hence, the map in (3.7) is p to 1, so we have $\#[B] = \frac{q^{[c/p]}}{p}$. Since $\#\mathfrak{H}_{P,c} = \#\mathbb{A}_{\deg(P)+c} - \#\mathbb{A}_c = q^c(q-1)(q^{\deg(P)} - 1)$, we have

$$\#\tilde{\mathfrak{H}}_{P,c} = pq^{c-[c/p]}(q-1)(q^{\deg(P)} - 1).$$

THEOREM 3.3. *For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$, positive integer c with pc and $1 \le i \le p-1$, we have*

$$\lim_{m\to\infty} \frac{\sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{[B]\in\tilde{\mathfrak{H}}_{P,c}} L(s, \chi^i_{B/P})}{\tilde{I}_q(m,c)} = \zeta_{\mathbb{A}}(ps),$$

where $\tilde{I}_q(m,c) = pq^{c-[c/p]}(q-1)(q^m - 1)\#\mathcal{P}_m(\mathbb{A})$.

Proof. Let

$$h_{m;c}(s) = \frac{\sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{[B]\in\tilde{\mathfrak{H}}_{P,c}} L(s, \chi^i_{B/P})}{\tilde{I}_q(m,c)} - \zeta_{\mathbb{A}}(ps).$$

Since $L(s, \chi^i_{B/P})$ is a polynomial in q^{-s} of degree $2m + c - 1$ for $P \in \mathcal{P}_m(\mathbb{A})$ and $B \in \mathfrak{H}_{P,c}$, we have

$$\begin{aligned} &h_{m;c}(s) \\ (3.8)\qquad &= \frac{\sum_{n=0}^{2m+c-1} \sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{[B]\in\tilde{\mathfrak{H}}_{P,c}} \sigma_n^{(i)}(B/P) q^{-ns}}{\tilde{I}_q(m,c)} - \sum_{n=0}^{\infty} q^{n(1-ps)}. \end{aligned}$$

Put

$$h_{m,n;c} = \sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{[B]\in\tilde{\mathfrak{H}}_{P,c}} \sigma_n^{(i)}(B/P).$$

Then, we have

$$h_{m,n;c} = \frac{p}{q^{[c/p]}} \sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{N\in\mathbb{A}_n^+} \sum_{B\in\mathfrak{H}_{P,c}} \left\{\frac{B/P}{N}\right\}^i = \frac{p}{q^{[c/p]}} \sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{N\in\mathbb{A}_n^+} \tilde{T}^{(i)}_{N,P,c}.$$

If $P|N$, then $\tilde{T}^{(i)}_{N,P,c} = 0$ by definition. If PN and N is a p-th power, then $\tilde{T}^{(i)}_{N,P,c} = q^c(q-1)(q^m - 1)$. If PN and N is not a p-th power, by Corollary 2.4, we have $\tilde{T}^{(i)}_{N,P,c} = -\sum_{B\in\mathbb{A}_c}\{\frac{B}{N}\}^i$. Put $\alpha_{N,c} = \sum_{B\in\mathbb{A}_c}\{\frac{B}{N}\}^i$. If pn, since any $N \in \mathbb{A}_n^+$ will never be a p-th power, we have

$$h_{m,n;c} = -\frac{p}{q^{[c/p]}} \sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{N\in\mathbb{A}_n^+} \alpha_{N,c}. \tag{3.9}$$

If $p|n$, we have

$$\begin{aligned} h_{m,n;c} = &-\frac{p}{q^{[c/p]}} \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N:\text{not } p\text{-th power}}} \alpha_{N,c} \\ &+ pq^{c-[c/p]}(q-1)(q^m-1) \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N:p\text{-th power}}} 1. \end{aligned}$$

Since

$$\sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N:p\text{-th power}}} 1 = \sum_{N_1 \in \mathbb{A}_{n/p}^+, PN_1} 1 = \begin{cases} q^{\frac{n}{p}} & \text{if } \frac{n}{p} < m, \\ q^{\frac{n}{p}}(1-q^{-m}) & \text{if } \frac{n}{p} \geq m, \end{cases}$$

we have

(3.10)

$$\begin{aligned} &h_{m,n;c} \\ &= -\frac{p}{q^{[c/p]}} \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N:\text{not } p\text{-th power}}} \alpha_{N,c} + \begin{cases} q^{\frac{n}{p}} \tilde{I}_q(m,c) & \text{if } \frac{n}{p} < m, \\ q^{\frac{n}{p}}(1-q^{-m})\tilde{I}_q(m,c) & \text{if } \frac{n}{p} \geq m. \end{cases} \end{aligned}$$

By inserting (3.9) and (3.10) into (3.8) and rearranging the terms, we have

$$\begin{aligned} h_{m;c}(s) = &-q^{-m} \sum_{n=m}^{[\frac{2m+c-1}{p}]} q^{n(1-ps)} - \sum_{n=[\frac{2m+c-1}{p}]+1}^{\infty} q^{n(1-ps)} \\ &- \frac{1}{\tilde{I}_q(m,c)} \sum_{n=0}^{2m+c-1} \frac{p}{q^{[c/p]}} \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{\substack{N \in \mathbb{A}_n^+, PN \\ N:\text{not } p\text{-th power}}} \alpha_{N,c} q^{-ns}. \end{aligned}$$

For $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{p}$, as $m \to \infty$, we have

$$\begin{aligned} \left| q^{-m} \sum_{n=m}^{[\frac{2m+c-1}{p}]} q^{n(1-ps)} \right| &\leq q^{-m} \sum_{n=m}^{[\frac{2m+c-1}{p}]} q^{n(1-p\sigma)} \\ &\leq q^{-mp\sigma} \left(\left[\frac{2m+c-1}{p} \right] - m + 1 \right) \to 0 \end{aligned}$$

and

$$\left|\sum_{n=[\frac{2m+c-1}{p}]+1}^{\infty} q^{n(1-ps)}\right| \leq \sum_{n=[\frac{2m+c-1}{p}]+1}^{\infty} q^{n(1-p\sigma)} = \frac{q^{(1-p\sigma)([\frac{2m+c-1}{p}]+1)}}{1-q^{(1-p\sigma)}} \to 0.$$

Note that

$$\sum_{\substack{N\in\mathbb{A}_n^+, PN\\ N:\text{not } p\text{-th power}}} 1 = \#\mathbb{A}_n^+ - \sum_{N\in\mathbb{A}_n^+, P|N} 1 - \sum_{\substack{N\in\mathbb{A}_n^+, PN\\ N:p\text{-th power}}} 1,$$

where

$$\sum_{N\in\mathbb{A}_n^+, P|N} 1 = \begin{cases} 0 & \text{if } n < m,\\ q^{n-m} & \text{if } m \leq n \leq 2m+c-1.\end{cases}$$

Since c will be fixed and we will take $m \to \infty$, without loss of generality, we may assume $m > c$, so that $n \leq 2m+c-1 < 3m-1$. Since the proof of theorem for $p = 2$ is already given in [1], we will assume that p is odd, so that $n < pm$. If N is a p-th power and $P|N$, then $P^p|N$, so $pm \leq n$. Then, we have

$$\sum_{\substack{N\in\mathbb{A}_n^+, PN\\ N:p\text{-th power}}} 1 = \sum_{\substack{N\in\mathbb{A}_n^+\\ N:p\text{-th power}}} 1 = \begin{cases} 0 & \text{if } pn,\\ q^{\frac{n}{p}} & \text{if } p|n.\end{cases}$$

Hence, we have

$$\sum_{\substack{N\in\mathbb{A}_n^+, PN\\ N:\text{not } p\text{-th power}}} 1 = \begin{cases} q^n, & \text{if } n < m,\ pn,\\ q^n - q^{\frac{n}{p}}, & \text{if } n < m,\ p|n,\\ q^n - q^{n-m}, & \text{if } n \geq m,\ pn,\\ q^n - q^{n-m} - q^{\frac{n}{p}}, & \text{if } n \geq m,\ p|n.\end{cases} \tag{3.11}$$

For $\sigma = \mathrm{Re}(s) > \frac{1}{2}$, by using the fact that $|\alpha_{N,c}| \leq \#\mathbb{A}_c = (q-1)q^c$ and (3.11), we have

$$\left|\frac{1}{\tilde{I}_q(m,c)} \sum_{n=0}^{2m+c-1} \frac{p}{q^{[c/p]}} \sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{\substack{N\in\mathbb{A}_n^+, PN\\ N:\text{not } p\text{-th power}}} \alpha_{N,c} q^{-ns}\right|$$

$$\leq \frac{1}{\tilde{I}_q(m,c)} \sum_{n=0}^{2m+c-1} \frac{p}{q^{[c/p]}} \sum_{P\in\mathcal{P}_m(\mathbb{A})} \sum_{\substack{N\in\mathbb{A}_n^+, PN\\ N:\text{not } p\text{-th power}}} |\alpha_{N,c}| q^{-n\sigma}$$

$$\leq \frac{1}{q^m-1}\left(\sum_{n=0}^{2m+c-1} q^{n(1-\sigma)} - \sum_{\substack{n=0\\ p|n}}^{2m+c-1} q^{n(\frac{1}{p}-\sigma)} - q^{-m}\sum_{n=m}^{2m+c-1} q^{n(1-\sigma)}\right)$$

$$< \frac{1}{q^m-1}\sum_{n=0}^{2m+c-1} q^{n(1-\sigma)}.$$

If $\sigma = 1$, we have

$$\frac{\sum_{n=0}^{2m+c-1} q^{(1-\sigma)n}}{q^m-1} = \frac{2m+c}{q^m-1} \to 0$$

and if $\sigma \neq 1$, we have

$$\frac{\sum_{n=0}^{2m+c-1} q^{(1-\sigma)n}}{q^m-1} = \frac{1}{1-q^{-m}} \cdot \frac{q^{-m} - q^{m(1-2\sigma)+c(1-\sigma)}}{1-q^{(1-\sigma)}} \to 0$$

as $m \to \infty$. This completes the proof. □

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